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Adding flavor to Dijkgraaf-Vafa

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We study matrix models related via the prescription of Dijkgraaf and Vafa to supersymmetric gauge theories with matter in the fundamental. As in flavorless examples, measure factors of the matrix integral reproduce information about R-symmetry violation in the field theory. The models, similar to one studied by Kazakov, exhibit a large-M phase transition as the number of flavors is varied. This is the matrix model's manifestation of the end of asymptotic freedom.

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1. Introduction and summary

Dijkgraaf and Vafa have found a prescription to compute all of the holomorphic information in $\mathcal{N}=1$ gauge theories with classical gauge groups [1,2,3]. In the interest of understanding their proposal better, we will look at the matrix model that they prescribe for $\mathcal{N}=2$ gauge theory with N_f fundamental hypers mass-deformed to $\mathcal{N}=1$ by a tree level superpotential for the adjoint chiral field Φ, and masses for the squarks. These field theories, studied in [4], exhibit fascinating behavior as a function of the number of flavors.

Such models arise in string theory in a number of ways. The $U(N)$ theories arise by probing the "canonical example" of Dijkgraaf-Vafa with D5-branes on a noncompact curve of the Calabi-Yau (CY) [5]. Flavorful theories with real gauge groups arise on N D3-brane probes of D7-brane configurations [6,7,8,9]; this fact was recently exploited to good effect to add holes to the BMN worldsheet [10]. We will instead use it to poke holes in the random surfaces described by the Dijkgraaf-Vafa matrix integrals.

In particular, the theories with $N_f \leq N$ (with $Sp(2N)$ gauge group and an extra hypermultiplet in the antisymmetric tensor representation) arise on N D3-brane probes of a resolved D_4 singularity of F-theory. The theory with $N_f = N$ massless flavors is obtained by creating the D_4 singularity; in this case the dilaton is constant and the gauge theory is finite. This brings us to the question we would like to answer: How is this dependence on N_f/N manifested in the Dijkgraaf-Vafa matrix model?

We are therefore led to consider a matrix integral of the form

$$Z(g_k, m_i, M) = \int d\Phi dQ d\tilde{Q} \exp\left(-W_0(\Phi) + \tilde{Q}_i \Phi Q^i - \sum_i \tilde{Q}_i Q^i m_i\right) \tag{1.1}$$

where

$$W_0(\Phi) = \sum_k g_k \mathrm{tr}\ \Phi^k,$$

Φ is a complex $M \times M$ matrix, Q is a complex $M \times M_f$ rectangle, and $\tilde{Q}$ is a complex $M_f \times M$ rectangle. As in the work of Dijkgraaf and Vafa, these are to be thought of as line integrals over matrices of complex numbers.

Without using very much technology, we can study in detail the model which arises upon integrating out the fundamentals. This generates a $\ln(m - \Phi)$ potential for Φ. The model with this addition to the potential can still be solved by the method of BIPZ [11](and actually was studied in this way as a model for open strings by Kazakov [12]). The coefficient of the log is $\gamma \equiv N_f/N$, and we will show that the cuts close up when

γ is chosen so that the gauge theory can be conformal. Related by supersymmetry to this statement is the fact that the R-symmetry becomes non-anomalous for this choice of N_f/N. This fact can be detected in the matrix model through the dependence of the measure on γ.

M's and N's

One of the more mysterious aspects of the Dijkgraaf-Vafa prescription, at least to the author, is the disjunction between the number of colors N of the gauge theory, and the number of colors M of the matrix model (which plays the role of the glueball superfield). The addition of N_f flavors to the gauge theory only complicates this issue. We will find it necessary to introduce a number of flavors M_f in the matrix model which is again not the same as the corresponding number in the gauge theory. We will, however, identify the ratio

$$\gamma \equiv \frac{N_f}{N} = \frac{M_f}{N};$$

This will be the parameter of interest in our study of the matrix model. We take this as part of the prescription, but (thinking of γ as the weight with which holes in the random surface contribute) one which is again motivated by topological string duality [13].

Related work

Matrix models with fundamentals in this context are mentioned in a footnote in [1]. They also make an appearance in the very recent [14,15]. Other work on understanding and extending the Dijkgraaf-Vafa proposal includes [16,17,18,19,20,21,22,23,24].

2. Flavorful matrix models

To begin, we write down the matrix integral with naive couplings and fields marked with hats:

$$Z = \int d\hat{\Phi} d\hat{Q} d\hat{\tilde{Q}} \exp\left(-\hat{W}_0(\hat{\Phi}) + \hat{\tilde{Q}}_i \hat{\Phi} \hat{Q}^i - \sum_i \hat{\tilde{Q}}_i \hat{Q}^i \hat{m}_i \right) \tag{2.1}$$

with

$$\hat{W}_0(\hat{\Phi}) \equiv \hat{g}_1 \mathrm{tr}\ \hat{\Phi} + \hat{g}_2 \mathrm{tr}\ \hat{\Phi}^2 + \dots .$$

These hatted fields and couplings will be related to those which should have finite large-M limits (which lack hats) by an M-dependent rescaling. These hatless variables are chosen below so that there is a well-peaked saddle point of the Φ integral, the location of which is M-independent. We will observe that the precious

$$\frac{1}{2}M^2 \ln M$$

term in the matrix model free energy, which is derived from the inverse volume of the matrix model gauge group, can also be detected by such a propitious field rescaling.

2.1. Integrating out the flavor

For the moment, we are interested in the regime of couplings where the quark masses, m_i, are much bigger than the bare mass of the adjoint in $W_0(\Phi)$. In this regime, we integrate out the fast Q modes at fixed Φ to get an effective potential for Φ. The integrals over Q_i are M_f independent gaussian integrals. This gives

$$\begin{aligned} Z &= \int d\hat{\Phi} e^{-\hat{W}_0(\hat{\Phi})} \prod_{i=1}^{M_f} \det_{ab}\left(\hat{\Phi}^a_b - \hat{m}_i \delta^a_b\right) \\ &= \int d\hat{\Phi} \exp\left(-\hat{W}_0(\hat{\Phi}) + \sum_{i=1}^{M_f} \mathrm{tr}\ln(\hat{\Phi} - \hat{m}_i 1)\right) \end{aligned} \tag{2.2}$$

Introducing $\gamma \equiv M_f/M$, and for simplicity setting all of the masses equal to m, this is

$$Z = \int d\hat{\Phi}\ \exp\left(-\hat{W}_0(\hat{\Phi}) + \gamma \mathrm{tr}\ln(\hat{\Phi} - \hat{m})\right) \tag{2.3}$$

A matrix integral very similar to (2.3) was studied by Kazakov [12] as a discretization of an open string worldsheet[1]. In this model, the counterpart of γ is the weight accompanying a hole insertion after summing over flavors. The logarithmic potential was chosen to reproduce a sum over discretizations of the worldsheet boundaries, with equal weight for arbitrary numbers of segments of the boundary.

Other than a relabeling of couplings, the difference between our model and that of Kazakov is that the logarithmic potential term of [12] is

$$\mathrm{tr}\ln(m - \varphi^2).$$

This is the potential that would arise if the $\mathcal{N} = 2$ superpotential were $\tilde{Q}\Phi^2 Q$ instead of $\tilde{Q}\Phi Q$. The field redefinition $\Phi = \varphi^2$ required to relate the two integrals directly introduces a jacobian factor which adds a term

$$\frac{1}{2}\mathrm{tr}\ln\Phi$$

to the potential. From the calculation (2.2) above we see that this is the same as the effect of adding $M/2$ *massless* hypers. We will find it convenient to solve the integral (2.3) directly. The qualitative behavior we find is the same as that found in [12].

[1] A related model was also studied by Distler and Vafa [25].

2.2. The continuum

In order to proceed, diagonalize the matrix Φ as $\Phi = \mathcal{U}\mathcal{D}\mathcal{U}^\dagger$ with

$$\mathcal{D} \equiv \mathrm{diag}(\lambda_1, \ldots, \lambda_M).$$

These eigenvalues are normalized as in [11]. By the magic of logarithms, the integrand of (2.3) does not depend on the angular $\mathcal{U}$ variables. Their integration produces the Vandermonde determinant

$$\Delta(\lambda) = \prod_{a<b} (\lambda_a - \lambda_b)^2.$$

The integral becomes

$$Z = \int \prod_a d\lambda_a \Delta(\lambda) \exp \sum_{a=1}^{M} \left(-\hat{W}_0(\lambda_a) + \sum_{i=1}^{M_f} \ln(\lambda_a - \hat{m}_i) \right) \tag{2.4}$$

At this point, it is convenient to introduce a continuum in the space of colors. Let

$$\lambda(a/M) = \sqrt{M}\lambda(\tilde{a}), \qquad 1 = \int_0^1 d\tilde{a} = \frac{1}{M}\sum_{a=1}^{M}.$$

The eigenvalue density

$$\rho(\mu) = 2\frac{d\tilde{a}}{d\lambda}$$

is normalized to

$$\int d\mu \rho(\mu) = 2. \tag{2.5}$$

Large M scaling

We now introduce the promised variables without hats:

$$\begin{gathered} \hat{g}_2 = g_2, \quad \hat{g}_3 = \frac{g_3}{\sqrt{M}}, \quad \ldots, \hat{g}_k = \frac{g_k}{M^{k/2-1}} \\ \hat{m}_i = \sqrt{M} m_i \\ \hat{\Phi} = \sqrt{M}\Phi, \quad \hat{Q} = M^{-1/4} Q, \quad \hat{\tilde{Q}} = M^{-1/4}\tilde{Q}. \end{gathered} \tag{2.6}$$

Plugging these into (2.1), we find that the resulting λ integral is of the form

$$Z = \int D\lambda(\tilde{a}) \exp\left(M^2 \mathcal{I}[\lambda] + \mathcal{C}(M)\right) \tag{2.7}$$

with $\mathcal{C}(M)$ independent of λ and

$$\begin{aligned}\mathcal{I} \equiv & \int_0^1 d\tilde{a} \int_0^1 d\tilde{b} \ln(\lambda(\tilde{a}) - \lambda(\tilde{b})) - \int_0^1 d\tilde{a} W_0(\lambda(\tilde{a})) - \gamma \int_0^1 d\tilde{a} \ln(\lambda(\tilde{a}) - m) \\ = & \frac{1}{4} \int d\lambda \int dz \rho(\lambda)\rho(z) \ln(\lambda - z) - \frac{1}{2} \int d\lambda \rho(\lambda) W_0(\lambda) - \frac{\gamma}{2} \int d\lambda \rho(\lambda) \ln(\lambda - m)\end{aligned} \tag{2.8}$$

Here $W_0(\lambda) = g_1\lambda + g_2\lambda^2 + \ldots$. The crucial feature of (2.7) is that $\mathcal{I}[\lambda]$ is independent of M. This is the normalization used by Dijkgraaf and Vafa.

Now we return to the "constant g-independent term" [11] $\mathcal{C}(M)$. This field- and coupling-independent term was not relevant for previous applications of matrix integrals. It is

$$\begin{aligned}\mathcal{C}(M) &= \frac{1}{2} M^2 \ln M - \frac{1}{4} M_f M \ln M \\ &= (2-\gamma)\frac{1}{4} M^2 \ln M.\end{aligned} \tag{2.9}$$

This reproduces the leading M-dependence of the log of the inverse volume of $U(M)$ [26] in the field normalization we are using. Further, it provides the "entropy factor" arising from the flavor integrals[2]. The Dijkgraaf-Vafa prescription relates the matrix model free energy to the prepotential of the gauge theory. This term leads to the following contribution to the effective superpotential of the gauge theory:

$$W_{eff} = (2-\gamma) N S \ln \frac{S}{\Lambda^3}. \tag{2.10}$$

The prefactor of this Veneziano-Yankielowicz superpotential is proportional to the anomaly in the $U(1)_R$ current of the field theory. The introduction of N_f flavors in the fundamental modifies this from $2N$ to $N(2-\gamma)$. It is gratifying that this is reproduced by the simple matrix integral.

Solution at large M

In terms of the variables normalized to have a finite large-M limit, the saddle point equation is

$$W_0'(\lambda) + \frac{\gamma}{\lambda - m} = \fint d\mu \frac{\rho(\mu)}{\lambda - \mu} \tag{2.11}$$

Rewrite this equation as

$$W_0'(\lambda) = \fint d\mu \frac{\rho_0(\mu)}{\lambda - \mu} \tag{2.12}$$

[2] I am grateful to Nissan Itzhaki for comments on this point.

where

$$\rho_0(\mu) \equiv \rho(\mu) - \gamma\delta(\mu - m), \tag{2.13}$$

or more generally in the case of arbitrary masses

$$\rho_0(\mu) = \rho(\mu) - \gamma \frac{1}{M_f} \sum_{i=1}^{M_f} \delta(\mu - m_i).$$

Therefore the eigenvalue density ρ_0 satisfies the same integral equation as that of the theory *without* flavors, with the modified boundary condition

$$\int d\mu \rho_0(\mu) = \int d\mu \rho(\mu) - \int d\mu \sum_{i=1}^{M_f} \frac{\gamma}{M_f} \delta(\mu - m_i) = 2 - \gamma.$$

This integral is to be performed over the real μ line.

For simplicity, let us consider the case of a cubic superpotential, in the case of a single cut, *i.e.* choose the potential W_0 to have a single critical point. Placing the cut at $[2a, 2b]$, the BIPZ solution for the resolvent

$$\omega_0(\lambda) = \int_{2a}^{2b} d\mu \frac{\rho_0(\mu)}{\mu - \lambda} \tag{2.14}$$

of the theory without flavors determines the solution of (2.11). This is [11]

$$\omega_0(z) = 2g_2 z + 3g_3 z^2 - (2g_2 + 3g_3(a+b) + 3g_3 z)\sqrt{(z-2a)(z-2b)}. \tag{2.15}$$

The conditions determining the positions a, b of the ends of the cut are determined by the behavior of (2.15) at $z \to \infty$. For the model with flavor, these are

$$3g_3(b-a)^2 + 2(a+b)(2g_2 + 3g_3(a+b)) = 0 \tag{2.16}$$

$$\frac{1}{2}(b-a)^2(2g_2 + 6g_3(a+b)) = 2 - \gamma. \tag{2.17}$$

These differ from equations (46) of [11] only by the replacement $2 \mapsto 2 - \gamma$ in the second condition. Note that they do not depend on m.

Assume there is a stable vacuum at the origin, and place all of the eigenvalues there, *e.g.* consider $g_k = 0, k \geq 3$, $g_2 > 0$. From (2.16) and (2.17) we immediately see that when $\gamma \to 2$, $b - a \to 0$. That is, the cut closes up. Beyond $\gamma = 2$, the cut at the origin moves into the complex plane. In the context of a Hermitian matrix integral [12] this was interpreted as a large-M phase transition beyond which the theory lacked a stable solution. However, when the saddle point we are studying is that of a holomorphic line integral, this is not so catastrophic. In fact, the cut surrounding the unstable extrema of W always extend into the imaginary direction of λ.

In the matrix models for confining gauge theories, the size of the cuts which hold the eigenvalues goes like the IR scale Λ of the field theory. The closing of the cuts is a signal that the theories become scale invariant, and then no longer asymptotically free as N_f passes through $2N$.

2.3. Towards the Seiberg-Witten curve

In this subsection we take a few steps towards reproducing the Seiberg-Witten (SW) curve of the gauge theory from the loop equation of the matrix model.

According to Dijkgraaf-Vafa, the free energy of the matrix model determines the effective glueball superpotential of the field theory via

$$W_{eff} = \sum_i N_i \frac{\partial \mathcal{F}_0(S)}{\partial S_i} - 2\pi i \tau S. \tag{2.18}$$

From the form of the solution for the eigenvalue density (2.13), and from the expression (2.8) for the free energy in terms of ρ, we see that the free energy of the model with flavor is

$$\begin{aligned}\mathcal{F}_0(S,\gamma) = \frac{1}{4}\int d\lambda \int dz\, (\rho_0(\lambda) + \gamma\delta(\lambda - m))\, (\rho_0(z) + \gamma\delta(z - m)) \ln(\lambda - z) \\ - \frac{1}{2}\int d\lambda\, (\rho_0(\lambda) + \gamma\delta(\lambda - m))\, W_0(\lambda) - \frac{\gamma}{2}\int d\lambda\, (\rho_0(\lambda) + \gamma\delta(\lambda - m)) \ln(\lambda - m).\end{aligned}$$

Note that ρ_0 depends on γ through its dependence on the locations of the cuts (Eqn. (2.17)), so this is not merely $\mathcal{F}_0(S, \gamma = 0) + \ldots$. We note that evaluating the δ functions gives a divergence of the form $\frac{\gamma^2}{2}\ln(m - m)$, which appears to be independent of the S_i.

Loop equation

At large M, the resolvent

$$\omega_0(x) = \int d\lambda \frac{\rho_0(\lambda)}{x - \lambda}$$

of the $\gamma = 0$ theory satisfies an algebraic equation of the form

$$\omega_0(x)^2 + \frac{1}{S}\omega_0(x)W_0'(x) + \frac{1}{4S^2} f_0(x) = 0. \tag{2.19}$$

Here $S = g_s M$ is the 't Hooft coupling/glueball field, and $f_0(x)$ is a polynomial in x, which should be thought of as a function of S_i, the number of eigenvalues in each cut. Please note that this polynomial f_0 differs from the one in the $\gamma = 0$ solution through its dependence on the locations of the cuts. The resolvent of the theory with flavor is related to ω_0 by the addition of a pole term

$$\omega(x) = \omega_0(x) + \frac{\gamma}{x - m}. \tag{2.20}$$

Extracting the S_i-dependence from the free energy $\mathcal{F}_0$, and minimizing the resulting W_{eff} (2.18) with respect to S_i, the equation (2.19) (for W_0 of degree $N+1$) should result in the Seiberg-Witten curve

$$y^2 = \prod_{a=1}^{N} (x - \phi_a)^2 + \Lambda^{2N-N_f} (x - m)^{N_f}, \tag{2.21}$$

In the case without flavor, y arose as the singular piece of the resolvent, by completing the square in (2.19) [1]

$$y(x) = 2S\omega_0(x) + W_0'(x).$$

To reproduce the polynomial equation (2.21) given (2.20) will require a more involved change of variables; we leave this for the future.

3. Discussion and prospects

In this paper, we have focused on the regime of couplings where $m_\Phi << m_Q$ (though of course we can still perform the gaussian integral over Q in the other regime) where the interesting observables involve the adjoint field, as in the case without flavors. It will be interesting to try to compute observables involving meson and baryon operators

$$Z[M] = \int d\Phi dQ \exp\left(W_0(\Phi) - \tilde{Q}_i \Phi Q^i + \sum_i \tilde{Q}_i Q^i m_i \right)$$
$$\exp\left(\tilde{Q}_{ia} M^i_j Q^a_j + B^{i_1 \ldots i_N} \epsilon_{a_1 \ldots a_N} Q_{a_1 i_1} \ldots Q_{a_N i_N} + \tilde{B}^{i_1 \ldots i_N} \epsilon_{a_1 \ldots a_N} \tilde{Q}_{a_1 i_1} \ldots \tilde{Q}_{a_N i_N} \right)$$

These baryon sources exist for $N_f \geq N$, corresponding to a value of γ at which we have not yet detected any change in behavior of the matrix model.

Consider for the moment just a color-singlet meson source:

$$Z(M) = \int d\Phi dQ d\tilde{Q} e^{W_0(\Phi) - \tilde{Q}_i \Phi Q^i} \exp \tilde{Q}_{ia} M^i_j Q^{ja}. \tag{3.1}$$

Because M is gauge invariant, we can still do the Q integral, and obtain

$$Z(M) = \int d\Phi \frac{e^{W_0(\Phi)}}{\det\left(\Phi \otimes 1 - 1 \otimes M\right)} \tag{3.2}$$

One can take the point of view that the flavors arise from an $\mathcal{N}=2$ quiver with two nodes N and N_f where we turn off the coupling of the flavor gauge group. Then M arises from the adjoint chiral field in the $\mathcal{N}=2$ vectormultiplet for the flavor group. In this form, the integral (3.2) was studied in the recent [24]. From this perspective, the convenience of making a continuum of the space of flavors is less mysterious.

Beyond the transition

Consider the F-theory realization of these models. The flavor symmetry (which is $SO(8)$ in the critical case) is the gauge symmetry on the D7-branes. Increasing N_f/N beyond the critical value is achieved by adding more D7-branes to the D_4 singularity. This is possible without destroying the triviality of the canonical bundle, and one obtains in this way collections of D7-branes with the exceptional series of gauge groups of rank up to 8. The D3 probe theories are then field theories with exceptional flavor symmetry [*e.g.* 27,28,29,30,31].

It has become clear that the complex x-plane of Φ eigenvalues can be identified with the image of a fibration of a noncompact CY geometry. A cut which holds the eigenvalues in the large M solution is identified with the image in the x-plane of a three-cycle in this geometry (after a geometric transition induced by the flux generating $W_0(\Phi)$). It is therefore tempting to speculate that tuning γ past the critical value is the matrix model version of performing an extremal transition in the CY geometry, during which the three-cycle shrinks and one finds an even-dimensional cycle which can be resolved. The fact that a shrinking del Pezzo four-cycle in a CY realizes a theory with exceptional flavor symmetry leads to a clear candidate for the nature of this new direction.

Some remaining issues

1. Our calculations should extend to the case of real gauge groups, and in particular to the theories with extra tensor representations arising from D3-brane probes.
2. We have not yet succeeded in extracting the S_i-dependence of the matrix model free energy $\mathcal{F}_0(g_k, m, \gamma)$, nor in reproducing the expected SW curve (2.21).

3. "uv" completions of the Seiberg-Witten curve. As explained in [23], including more of the "fractional branes" of the CY singularity allows one to determine an embedding of the SW curve in a threefold of a form such as

$$uv = F(x, y).$$

This is important, for example, because it will allow one to identify the resolution involved in the extremal transition proposed above.
4. The critical field theories with $N_f = 2N$ exhibit S-duality. In a beautiful series of papers [18,19,20,24], the S-duality of the $\mathcal{N} = 4$ theory and its deformations has been found via the solution of the corresponding matrix integral in [32]. The $\gamma = 2$ solutions should also have modular behavior in τ.
5. Kazakov [12] computes "average numbers of holes" and "average lengths of holes" in the random surfaces, from the large M solution to the matrix integral. These observables exhibit more detailed critical behavior than we have discussed thus far as γ, m, g_k are varied. The transition to 'torn surfaces' with large holes likely has an interpretation in terms of the appearance of a Higgs branch in the gauge theory.

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